Useful calculation:

$$\left(\sin \frac{n\pi}{L} x, \sin \frac{m\pi}{L} x\right) = \int_0^L \sin \frac{n\pi}{L} x \sin \frac{m\pi}{L} x \, dx$$

[can be solved using trig identity

$$\boxed{\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))}$$

[you can derive this for yourself using

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}), \quad \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$]

$$= \boxed{\frac{1}{2} \int_0^L \cos \frac{(n-m)\pi}{L} x - \cos \frac{(n+m)\pi}{L} x \, dx} = 0$$

except if $n = m$ (integers)
by calculations on next page

$$= \frac{1}{2} \left[\frac{L}{(n-m)\pi} \sin \frac{(n-m)\pi}{L} x - \frac{L}{(n+m)\pi} \sin \frac{(n+m)\pi}{L} x \right]_0^L =$$

Last class:

$$\int_0^L \sin\frac{n\pi}{L}x \ \sin\frac{m\pi}{L}x\,dx = \begin{cases} L/2 & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$$

recall: we defined $(f,g) = \int_0^L f(x)g(x)\,dx$

$$\Rightarrow \left(\sin\frac{n\pi}{L}x, \sin\frac{m\pi}{L}x\right) = \begin{cases} L/2 & n=m \\ 0 & n\neq m \end{cases}$$

Recall from Linear algebra:

If $u_1, u_2, \ldots, u_d$ in $\mathbb{R}^d$ nonzero vectors s.t. $(u_n, u_m) = u_n \cdot u_m = 0$ for $n\neq m$

and $v \in \mathbb{R}^d$

$$\Rightarrow v = \sum_{n=1}^{d} B_n u_n \quad \text{where } B_n = \frac{(v, u_n)}{(u_n, u_n)}$$

Same formula works for infinite dim. vector spaces

here: V = all integrable functions on interval $[0, L]$

$$\boxed{u_n = \sin \frac{n\pi}{L} x}$$

<u>Theorem</u> Let f be continuous function on $[0, L]$, $\begin{matrix} f(0) = 0 \\ = f(L) \end{matrix}$

$$\Rightarrow \boxed{f(x) = \sum_{n=1}^{\infty} B_n u_n = \sum B_n \sin \frac{n\pi}{L} x}$$

where $B_n = \frac{2}{L} \int f(x) \sin \frac{n\pi}{L} x \, dx$

$$\frac{2}{L} = \frac{1}{(u_n, u_n)}$$

<u>Remark</u>: theorem also works for not necessarily continuous functions

This was last missing piece for solving

(PDE) $\qquad \dfrac{\partial u}{\partial t} = h \dfrac{\partial^2 u}{\partial x^2}$

(BC) $\qquad u(0,t) = 0 = u(L,0)$

(IC) $\qquad u(x,0) = f(x)$ $\qquad$ given function f

Strategy to solve:

I Calculate product solutions $u(x,t) = G(t)\phi(x)$

ⓐ separate variables

$$\frac{G'(t)}{k\,G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

ⓑ solve two ODE's $\quad G'(t) = -\lambda k\, G(t)$

and $\quad \phi''(x) = -\lambda\, \phi(x)$

ⓒ use boundary conditions to determine possible values of λ (in our example $\lambda = \frac{n^2\pi^2}{L^2}$)

__Def.__ λ is called an eigenvalue of our PDE with given boundary conditions

II (a) For each eigenvalue λ find corresponding solution $\phi_\lambda(x)$ of ODE $\phi''(x) = -\lambda \phi(x)$

(in our example: $\lambda = \frac{n^2\pi^2}{L^2}$

$$\phi_\lambda = \sin \frac{n\pi}{L} x \quad)$$

(b) Find Fourier expansion of $f(x)$ in terms of eigenfunctions ϕ_λ

in our example: $f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$

where $B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$

(c) Solution: $\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \, e^{-\left(\frac{n\pi}{L}\right)^2 k t}}$

Concrete example:

$$f(x) = 100 \quad \text{for } 0 < x < 100$$

$$f(0) = 0 = f(100)$$

<u>solution</u>: have already shown:

can expand $f(x) = \sum B_n \sin \frac{n\pi}{L} x$

$\Rightarrow$ get solution

$$u(x,t) = \sum B_n \sin \frac{n\pi}{L} x \, e^{-\left(\frac{n\pi}{L}\right)^2 k t}$$

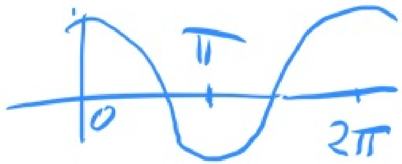
only thing left: calculate coefficients B_n

$$B_n = \frac{2}{L}\int_0^L \underset{\substack{\| \\ f(x)}}{100} \cdot \sin\frac{n\pi}{L}x\,dx$$

$$= \frac{200}{\cancel{L}}\,\frac{\cancel{L}}{n\pi}\left(-\cos\frac{n\pi}{L}x\right)\Bigg|_{x=0}^{x=L}$$

$$= \frac{200}{n\pi}\left(-\cos\frac{n\pi}{\cancel{L}}\cdot\cancel{L} + \underbrace{\cos\frac{n\pi}{L}0}_{=1}\right)$$

$$= \frac{200}{n\pi}\left(1 - \underbrace{\cos n\pi}_{=(-1)^n}\right) = \frac{200}{n\pi}\left(1-(-1)^n\right)$$



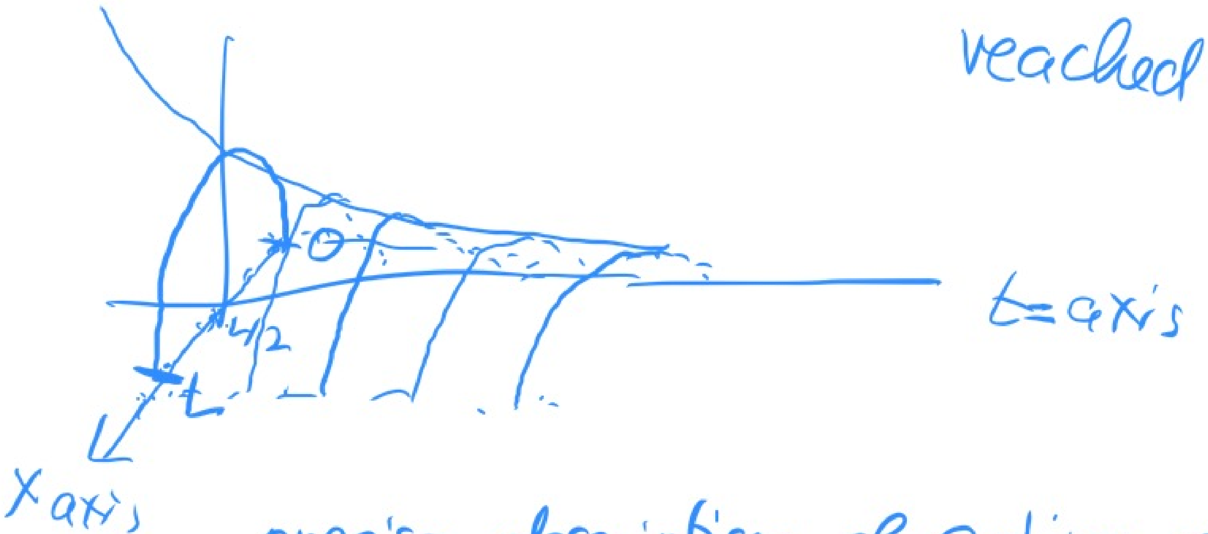
$$= \begin{cases} \frac{400}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$w(x,t) \approx \frac{400}{\pi} \sin\frac{\pi}{L}x \; e^{-\left(\frac{\pi}{L}\right)^2 k t}$$

see picture in book

max. amplitude at time $t = \frac{400}{\pi} e^{-\left(\frac{\pi}{L}\right)^2 k t}$

reached at $x = \frac{L}{2}$



precise description of cooling off of a rod with initial temperature $100°$ with constant boundary temp $0°$